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Solution of plane seepage problems for a multivalued seepage law when there is a point source☆

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ABSTRACT

The steady seepage of an incompressible fluid in a uniform porous medium, occupying an arbitrary bounded two-dimensional region, when there is a point source present is considered. Part of the boundary of the region is free, while the remaining part is impermeable for the fluid. It is assumed that the function defining the seepage law is multivalued and has a linear increase at infinity. A generalized formulation of the problem is proposed in the form of a variational inequality of the second kind. An approximate solution of the problem is obtained by an iterative splitting method, which enables approximate values of both the solution itself (the pressure) and its gradient to be found. Analytic expressions describing the boundaries of the region where the modulus of the pressure gradient takes a constant value are obtained for model problems of a line of bore holes. Numerical experiments are carried out for model problems, which confirm the effectiveness of the proposed method. Good agreement is observed between the results of calculations obtained analytically and by approximate methods.

Problems of the seepage of an incompressible fluid in a uniform porous medium for a multivalued seepage law with a limited gradient were considered in Ref. 1. Here, when the functions, defining the seepage law, and the seepage regions have a fairly simple form, exact solutions were obtained for a number of model problems with point sources (see, for example, Refs 2-4 and the references given there). In the case of an arbitrary seepage region, a singularity related to the presence of a source was only taken into account for finite-dimensional approximations of the problems (see, for example, Ref. 5). A mathematical formulation of the problem with a continuous law when a point source was present was given in Ref. 6 and an existence theorem was proved. A mathematical formulation of the problem was presented in Ref. 7 in the case of a multivalued law in the form of a variational inequality and its solvability was established.

An iterative splitting method was proposed in Ref. 8 for solving variational inequalities of the second kind and its convergence was investigated. This method is used below for the seepage problem considered. It enables approximate values of both the pressure and the pressure gradient to be obtained. It is verified that the conditions for the method to converge for variational inequalities as it applied to the seepage problem are satisfied. The main difficulty in using the iterative splitting method[8] is solving minimization problems, which arise in each iteration. However, the minimization problem can be solved in explicit form for the seepage problem, so that each step of the method is in fact reduced to solving a boundary-value problem for Laplace's equation.

1. Formulation of the problem

We will consider the seepage of an incompressible fluid in a bounded plane region Ω with a Lipschitz-continuous boundary

$$\Gamma = \Gamma_1 \bigcup \Gamma_2, \quad \Gamma_1 \bigcap \Gamma_2 = \varnothing, \quad \mathrm{mes}\Gamma_1 > 0$$

Part of the boundary Γ_1 is free, and the fluid pressure on it is equal to zero, while the part Γ_2 is an impermeable solid wall. We will assume that, at the internal point x^* of the region Ω, there is a point source of constant intensity $q > 0$. It is required to determine the steady

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fields of the pressure p and of the velocity $v = \{v_1, v_2\}$ of the fluid, satisfying the fluid mass balance equation

$$\mathrm{div}\, \upsilon = q\delta(x - x^*), \quad x \in \Omega \tag{1.1}$$

with a multivalued seepage law

$$-\upsilon \in \frac{g_\vartheta(|\nabla p|)}{|\nabla p|}\nabla p, \quad x \in \Omega \tag{1.2}$$

(the seepage law is written in the form of an inclusion, and not an equality, since g_ϑ is a multivalued function) and the corresponding boundary conditions

$$p(x) = 0, \quad x \in \Gamma_1, \quad (\upsilon, n) = 0, \quad x \in \Gamma_2 \tag{1.3}$$

where $\mathbf{n}$ is the unit vector of the outward normal to Γ_2.

We will assume that the multivalued function g_ϑ can be represented in the form

$$g_\vartheta(\xi) = g_0(\xi) + \vartheta H(\xi - \gamma)$$

where $\vartheta \geq 0$, $\gamma \geq 0$ are specified constants, and the multivalued function H and the single-valued function g_0 are given by the relations

$$H(\xi) = \begin{cases} 0, & \xi < 0 \\ [0, 1], & \xi = 0, \\ 1, & \xi > 0 \end{cases} \quad g_0(\xi) = \begin{cases} 0, & \xi < \gamma \\ g^*(\xi - \beta), & \xi \geq \gamma \end{cases}$$

the function g^*: $[0, +\infty] \to R^1$ satisfies the conditions

$$g^*(0) = 0, \quad g^*(\xi) > g^*(\zeta), \quad \forall \xi > \zeta > 0 \tag{1.4}$$

and constants $k > 0$, $\xi^* \geq 0$, $L > 0$ exist such that

$$g^*(\xi^*) \geq k\xi^*, \quad g^*(\xi) - g^*(\zeta) \geq k(\xi - \zeta), \quad \forall \xi \geq \zeta \geq \xi^* \tag{1.5}$$

$$|g^*(\xi) - g^*(\zeta)| \leq L|\xi - \zeta|, \quad \forall \xi, \zeta \geq 0 \tag{1.6}$$

There are examples of functions which satisfy the above conditions (see, for example, Refs 2 and 3).

We will define the operator G: $R^2 \to R^2$ in terms of the function g_0 as follows:

$$G(y) = \begin{cases} 0, & y = 0 \\ |y|^{-1} g_0(|y|) y, & y \neq 0 \end{cases}$$

By analogy with Ref. 6 we will formulate the following generalized form of problem (1.1)–(1.3).

We will put

$$V_m = \left\{\eta \in W_m^{(1)}(\Omega) : \eta(x) = 0, x \in \Gamma_1\right\}, \quad m = 1, \ 2$$

$$C_1^\infty = C^\infty(\bar{\Omega}) \cap V_1$$

We will mean by the solution of problem (1.1)–(1.3) a function $p \in V_1$, which is the solution of the variational inequality

$$q\eta(x^*) \leq \int_\Omega (G(\nabla p(x)), \nabla\eta(x))dx + F(\eta + p) - F(p), \quad \forall \eta \in C_1^\infty \tag{1.7}$$

where

$$F(\eta) = \vartheta \int_\Omega \mu(|\nabla\eta(x)| - \gamma)dx, \quad \mu(\xi) = \xi^+ = (\xi + |\xi|)/2$$

Following the previous approach,[6] we will consider the problem of finding the function $\omega_r \in \overset{\circ}{W}{}_1^{(1)}(B_r(x^*))$, which is the solution of the integral identity

$$\int_{B_r(x^*)} (G(\nabla\omega_r(x)), \nabla\eta(x))dx = q\eta(x^*), \quad \forall \eta \in C_0^\infty(B_r(x^*)) \tag{1.8}$$

where

$$B_r(x^*) = \{x \in R^2 : |x - x^*| < r\}$$

As with the results obtained previously,[2,6] we will present an explicit form of the solution of problem (1.8). We will denote the function inverse to g^* by h^*. It follows from conditions (1.4)–(1.6) that such a function h^* exists and is continuous. We will define the function h by the formula $h(\xi) = h^*(\xi) + \xi^*$, and also the function

$$p_r : (0, r] \to R^1, \quad p_r(s) = \int_s^r h\left(\frac{q}{2\pi\xi}\right) d\xi$$

It was shown in Ref. 6 that the function

$$w_r : B_r(x^*) \to R^1, \quad w_r(x) = p_r(|x - x^*|)$$

is the solution of problem (1.8). Then the following estimate holds

$$|\nabla w_r(x)| \le q/(2\pi k|x - x^*|) + 2\xi^* \tag{1.9}$$

We will choose the quantity r to be sufficiently large so that $\Omega \subseteq B_r(x^*)$. Since x^* is an internal point of the region Ω, we obtain $\varepsilon > 0$, for which the following inclusion is satisfied

$$\Gamma \subset \Omega_\varepsilon = B_r(x) \backslash B_\varepsilon(x^*)$$

It follows from inequality (1.9) that $w_r \in W_2^{(1)}(\Omega_\varepsilon)$, and so, its trace on Γ_1 is defined, and hence we obtain the function $w_\Gamma \in W_2^{(1)}(\Omega)$ such that

$$w_\Gamma(x) = -w_r(x), \quad x \in \Gamma_1 \tag{1.10}$$

An example is the function defined by the formula

$$w_\Gamma(x) = \begin{cases} -w_r(x), & x \in \Omega \backslash B_\varepsilon(x^*) \\ -p_r(\varepsilon), & x \in B_\varepsilon(x^*) \end{cases}$$

The solution of problem (1.7) will be sought in the form $p = w_r + w_\Gamma + w$, where $w \in V_2$ is an unknown function. Since $C_1^\infty \subseteq C_0^\infty(B_r(x^*))$, then, taking identity (1.8) into account, problem (1.7) reduces to finding the function $w \in V_2$, which is the solution of the variational inequality

$$\begin{aligned} &\int_\Omega (G(\nabla(w_r + w_\Gamma + w)) - G(\nabla w_r), \nabla(w + \eta) - \nabla w) dx + \\ &+ F(w_r + w_\Gamma + w + \eta) - F(w_r + w_\Gamma + w) \ge 0 \quad \forall \eta \in C_1^\infty \end{aligned} \tag{1.11}$$

where $(\cdot,\cdot)_V$ is the scalar product in V_2.

By analogy with the previous approach[6], it is easy to verify that the form

$$a(w, \eta) = \int_\Omega (G(\nabla(w_r + w_\Gamma + w)) - G(\nabla w_r), \nabla \eta dx, \quad w, \eta \in V_2$$

is linear and bounded on V_2 with respect to the second argument, and so it generates the operator A: $V_2 \to V_2$

$$(Aw, \eta)_V = a(w, \eta)$$

Hence problem (1.11) can be written in the form of the following operator variational inequality

$$(Aw, \eta - w)_V + \Psi(\eta) - \Psi(w) \ge 0, \quad \forall \eta \in V_2 \tag{1.12}$$

The functional $\Psi : V_2 \to R^1$ is defined by the formula

$$\Psi(w) = F(w_r + w_\Gamma + w) = \vartheta \int_\Omega \mu(|\nabla(w_r + w_\Gamma + w)| - \gamma) dx \tag{1.13}$$

It was proved in Ref. 7 that when conditions (1.4)–(1.6) are satisfied the set of solutions of problem (1.12) is not empty, convex or closed.

It should be noted that relations (1.1)–(1.3) also describe the problem of determining the boundaries of limit equilibrium pillars of residual viscoplastic petroleum (see Ref. 10), i.e., lines on which $|\nabla p| = \gamma$.

2. Iteration method

Variational inequality (1.12), which characterizes problem (1.1)–(1.3), can be written in the form

$$(Aw, \eta - w)_V + \Upsilon(\Lambda\eta) - \Upsilon(\Lambda w) \ge 0, \quad \forall \eta \in V_2 \tag{2.1}$$

if we put $\Lambda = \nabla$ and write the functional Ψ in the form $\Psi = \gamma \circ \Lambda$, where the functional γ is defined on $Y = [L_2(\Omega)^2]$ by the formula

$$\Upsilon(p) = \vartheta \int_\Omega \mu(|p + \nabla(w_r + w_\Gamma)| - \gamma) dx$$

To solve variational inequality (2.1) we will use the iterative splitting method proposed in Ref. 8, which consists of the following. Suppose $\rho > 0$, Λ^*: $Y \to V_2$ is the operator conjugate to Λ, and w_0 is an arbitrary element on V_2, y_0, λ_0 from Y. For $n = 0, 1, 2, \ldots$, knowing y_n and λ_n, we put

$$w_{n+1} = w_n - \tau[Aw_n + \Lambda^* \lambda_n + \rho(w_n - \Lambda^* y_n)] \tag{2.2}$$

We then find y_{n+1} by solving the minimization problem

$$\Upsilon(y_{n+1}) + \frac{\rho}{2}\|y_{n+1}\|_Y^2 - (\rho\Lambda w_{n+1} + \lambda_n, y_{n+1})_Y \le$$
$$\le \Upsilon(z) + \frac{\rho}{2}\|z\|_Y^2 - (\rho\Lambda w_{n+1} + \lambda_n, z)_Y, \quad \forall z \in Y \tag{2.3}$$

where $\|\cdot\|_Y$ $(\cdot,\cdot)_Y$ is the norm and scalar product in Y. We finally put

$$\lambda_{n+1} = \lambda_n + \rho(\Lambda w_{n+1} - y_{n+1}) \tag{2.4}$$

Theorem. *Suppose* $0 < \tau_0 \le \tau < 2\sigma/(2\sigma\rho + 1)$, $\rho = 1/L$ *and the iteration sequences* $\{w_n\}_{n=0}^{+\infty}$, $\{y_n\}_{n=0}^{+\infty}$, $\{\lambda_n\}_{n=0}^{+\infty}$ *are constructed in accordance with relations (2.2)–(2.4). The sequence* w_n *then weakly converges in* V_2 *to a certain solution* w *of problem (2.1), and the sequence* y_n *converges weakly in* Y *to* Λw *as* $n \to +\infty$.

Proof. Following the approach used previously in Refs 6 and 7, we can verify that the functional Ψ is Lipschitz-continuous and convex, while the operator A is inversely strongly monotonic, i.e.,

$$(Aw - A\eta, w - \eta)_V \ge L^{-1}\|Aw - A\eta\|_V^2, \quad \forall w, \eta \in V_2$$

and coercive, and hence the assertion of the theorem follows from the results obtained previously.[8]

When using the proposed iteration process numerically the main difficulty is carrying out its second step, i.e., solving minimization problem (2.3). We define the following functional on Y

$$\Upsilon_\rho(z) = \Upsilon(z) + \frac{\rho}{2}\| z \|_Y^2$$

Then inequality (2.3) can be rewritten in the form of the inclusion

$$\rho\Lambda w_{n+1} + \lambda_n \in \partial\Upsilon_\rho(y_{n+1})$$

equivalent to the following[9]

$$y_{n+1} \in \partial\Upsilon_\rho^*(\rho\Lambda w_{n+1} + \lambda_n)$$

where γ_ρ^* is a functional conjugate to γ_ρ. For the seepage problem considered, we must calculate

$$\partial\Upsilon_\rho^* = \Upsilon_\rho^{*\prime}: \quad \Upsilon_\rho^{*\prime} z = |z|^{-1} h_\rho(|z|) z$$

where

$$h_\rho(\xi) = \begin{cases} \xi/\rho, & \xi \le r\gamma \\ \gamma, & \rho\gamma < \xi \le \rho\gamma + \vartheta \\ (\xi - \vartheta)/r, & \xi > \rho\lambda + \vartheta \end{cases}$$

Hence, since the calculations of λ_n are carried out using explicit formulae, each step of iteration process (2.2)–(2.4) reduces to solving problem (2.2), i.e., by virtue of the Riesz – Fisher theorem, it actually reduces to solving a boundary-value problem for Laplace's equation with corresponding boundary conditions.

We constructed finite element approximations for variational inequality (2.1) and we investigated their convergence.

3. Accurate characteristics of the solution of model problems

We will consider the problem of determining the boundaries of the regions Ω_γ, when the modulus of the pressure gradient is equal to a specified value γ in the case of a rectilinear chain of bore holes with a flow rate q, equally spaced a distance $2l$ from one another. By virtue of the symmetry of the problem we can confine ourselves to an element of the flow, representing the half-strip $\{0 \le x \le l, y \ge 0\}$ (see Fig. 1, where $X = x/l$, $Y = y/l$).

We will consider the case (Problem 1) when the function g_ϑ has the form (see, for example, Ref. 3)

$$g_\vartheta(\xi) = \begin{cases} \alpha\xi, & 0 \le \xi < \gamma \\ [\alpha\gamma, \gamma], & \xi = \gamma; \quad \alpha \in (0,1) \\ \xi - \gamma(1-\alpha), & \xi > \gamma \end{cases} \tag{3.1}$$

Here, obviously $\vartheta = \gamma(1 - \alpha)$.

In order to construct analytic expressions for the lines describing the boundaries of the region with constant modulus of the pressure gradient, we will introduce variable hodographs of the velocity v, θ, where θ is the angle between the velocity vector and the x axis. The

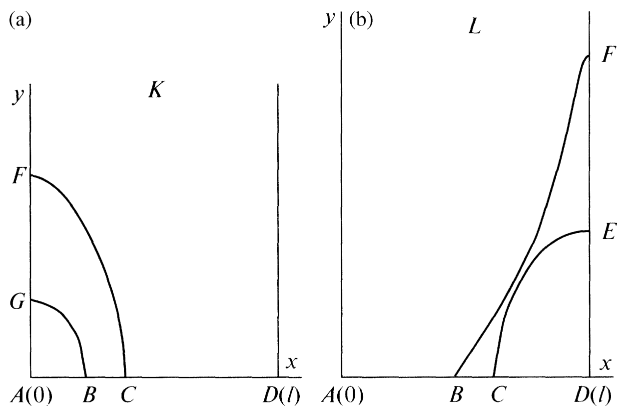

Fig. 1.

half-strip $\{0 \le \theta \le \pi/2, v \ge 0\}$ (see Fig. 2) in the plane of the variables v, θ corresponds to the half-strip in the physical plane. For the seepage law (1.2), (3.1) the stream function $\Psi(v, \theta)$ satisfies the linear equation[11]

$$\upsilon\frac{\partial}{\partial\upsilon}\left(\upsilon\frac{\partial\Psi}{\partial\upsilon}\right)+\frac{\partial^2\Psi}{\partial\theta^2}=0 \tag{3.2}$$

which it is required to solve in the regions (see Fig. 2)

$$D_1 = (0 \le \upsilon \le \alpha\gamma, 0 \le \theta \le \pi/2),$$
$$D_3 = (\beta \le \upsilon < +\infty, 0 \le \theta \le \pi/2)$$

The values of the function Ψ in regions D_1 and D_3 will be denoted by Ψ_1 and Ψ_3. In Fig. 2*a* and Fig. 2*b* the region

$$D_2 = (\upsilon = \alpha\gamma, \quad \upsilon = \gamma, \quad 0 \le \theta \le \pi/2)$$

corresponds to the regions *BCFG* and *BCEF* in Fig. 1*a* and Fig. 1*b* respectively. It follows from the results obtained previously,[10] that in these regions the streamlines are straight lines, the pressure along them varies linearly, while on the boundaries of the region D_2 the following conditions are satisfied when $0 \le \theta \le \pi$

$$\left.\left(\frac{\partial\Psi}{\partial\upsilon}\right)\right|_{\upsilon=\alpha\gamma}-\left.\left(\frac{\partial\Psi}{\partial\upsilon}\right)\right|_{\upsilon=\gamma}=\frac{1}{\gamma}\left(\frac{1}{\alpha}-1\right)\left.\left(\frac{\partial^2\Psi}{\partial\theta^2}\right)\right|_{\upsilon=\gamma}$$
$$\Psi(\alpha\gamma,\theta)=\Psi(\gamma,\theta) \tag{3.3}$$

The solution of the problem depends on the dimensionless parameters α and $Q = q/(4\gamma l)$. We will consider two cases: $Q \le \alpha < 1$ and $Q \ge 1$. When $Q \le \alpha < 1$ (Figs 1*a* and 2*a*) the following conditions are satisfied on the boundaries of regions D_1 and D_3:

$$\Psi(\upsilon,0)=0, \quad 0\le\upsilon\le\alpha\gamma, \quad \gamma\le\upsilon\le+\infty \tag{3.4}$$

$$\Psi(0,\theta)=0, \quad 0\le\theta\le\pi/2 \tag{3.5}$$

$$\Psi(\upsilon,\pi/2)=0, \quad 0\le\upsilon\le\gamma Q \tag{3.6}$$

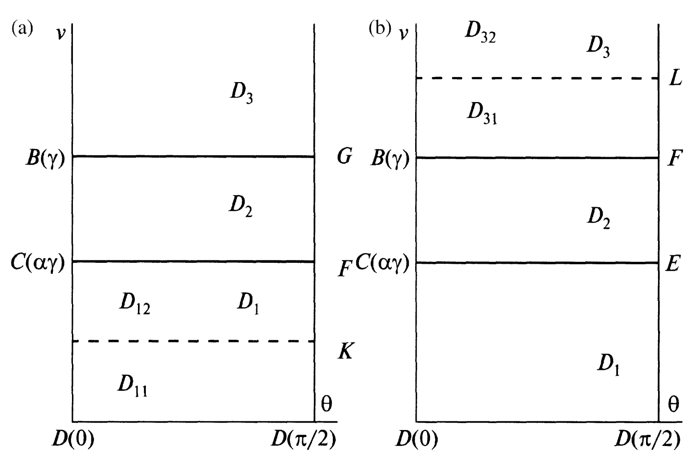

Fig. 2.

$$\Psi(\upsilon, \pi/2) = q/4, \quad \gamma Q \le \upsilon \le \alpha\gamma, \quad \gamma \le \upsilon < +\infty \tag{3.7}$$

$$\Psi(\upsilon, \theta) = (q/(2\pi))\theta, \quad \upsilon \to +\infty, \quad 0 \le \theta \le \pi/2 \tag{3.8}$$

The rectangle D_1 is divided by a section of the straight line $v = \gamma Q$ into two rectangles D_{11} and D_{12} (Fig. 2*a*), and the following "joining" conditions are satisfied in this section when $0 \le \theta \le \pi/2$

$$\Psi_{11}(\gamma Q, \theta) = \Psi_{12}(\gamma Q, \theta), \quad \left.\left(\frac{\partial \Psi_{11}}{\partial \upsilon}\right)\right|_{\upsilon=\gamma Q} = \left.\left(\frac{\partial \Psi_{12}}{\partial \upsilon}\right)\right|_{\upsilon=\gamma Q} \tag{3.9}$$

An analytic solution of problem (3.2)–(3.9) can be obtained by the method of separation of variables, using the procedure described in Ref. 12. We introduce the following notation

$$\tilde{\upsilon} = \frac{\upsilon}{\gamma Q}, \quad \kappa = \frac{Q}{\alpha}$$

The solution in the subregions D_{11}, D_{12} and D_3 is represented in the following form, respectively (everywhere henceforth summation is carried out over n from $n = 1$ to $n = \infty$)

$$\Psi_{11}(\upsilon, \theta) = \gamma l \sum A_n \tilde{\upsilon}^{2n} \sin 2n\theta$$

$$\Psi_{12}(\upsilon, \theta) = \gamma l \sum \left[B_{1n}(\kappa\tilde{\upsilon})^{2n} + B_{2n}\tilde{\upsilon}^{-2n} - 2B_{2n} \right] \sin 2n\theta$$

$$\Psi_3(\upsilon, \theta) = \gamma l \sum \left[C_n \tilde{\upsilon}^{-2n} - 2B_{2n} \right] \sin 2n\theta$$

where

$$A_n = \frac{(-1)^{n+1} Q}{\pi} \left[\frac{1}{n} + \frac{1}{a_n}\left(1 - \frac{1}{2n}\right)\kappa^{4n} \right]$$

$$B_{1n} = \frac{(-1)^{n+1} Q}{\pi a_n}\left(1 - \frac{1}{2n}\right)\kappa^{2n}, \quad B_{2n} = \frac{(-1)^n Q}{\pi n}$$

$$C_n = \frac{(-1)^n Q}{\pi n} \frac{1}{a_n(1-\alpha)}\kappa^{2n}, \quad a_n = n + \frac{1+\alpha}{2(1-\alpha)}$$

Reversion to the x, y flow plane is achieved using well-known transition formulae.[11] The coordinates $X_1(\theta)$, $Y_1(\theta)$ and $X_2(\theta)$, $Y_2(\theta)$ of the lines of constant value of the velocity BG ($|\nabla v| = \gamma$) and CF ($|\nabla v| = \alpha\gamma$) (Fig. 1*a*) have the form

$$\begin{aligned}
X_1(\theta) &= \frac{2}{\pi}\left[Q\cos\theta - \frac{\alpha}{1-\alpha}\sum \frac{f_n \cos(2n+1)\theta}{2n+1} \right] \\
Y_1(\theta) &= \frac{2}{\pi}\left[Q\sin\theta - \frac{\alpha}{1-\alpha}\sum \frac{f_n \sin(2n+1)\theta}{2n+1} \right] \\
X_2(\theta) &= \frac{1}{\pi}\left[\operatorname{arctg}\frac{2\kappa\cos\theta}{1-\kappa^2} - \sum f_n \cos(2n-1)\theta \right] \\
Y_2(\theta) &= \frac{1}{\pi}\left[\frac{1}{2}\ln\delta_0 + \sum f_n \sin(2n-1)\theta \right]
\end{aligned} \tag{3.10}$$

where

$$\delta_0 = \frac{1 + 2\kappa\sin\theta + \kappa^2}{1 - 2\kappa\sin\theta + \kappa^2}, \quad f_n = \frac{(-1)^{n+1}\kappa^{2n+1}}{a_n}$$

It can be verified that the series in (3.10) converge absolutely.

When $Q = \alpha = ½$ we have

$$x_1(0) = 1 - \frac{7}{3}\pi, \quad y_1(\pi/2) = \frac{5}{3}\pi, \quad x_2(0) = \frac{4}{3}\pi$$

When $Q \ge 1$ (Fig. 1*b* and 2*b*) conditions (3.4), (3.5) and (3.8) are satisfied, and

$$\Psi(\upsilon, \pi/2) = 0, \quad 0 \le \upsilon \le \alpha\gamma, \quad \gamma \le \upsilon \le \gamma Q \tag{3.11}$$

$$\Psi(\upsilon, \pi/2) = q/4, \quad \gamma Q \le \upsilon < \infty \tag{3.12}$$

The half-strip D_3 is divided by a section of the straight line $\upsilon = \gamma Q$ into a rectangle D_{31} and a half-strip D_{32} (Fig. 2b), and on this section

$$\Psi_{31}(\gamma Q, \theta) = \Psi_{32}(\gamma Q, \theta)$$
$$\left(\frac{\partial \Psi_{31}}{\partial \upsilon}\right)\Bigg|_{\upsilon=\gamma Q} = \left(\frac{\partial \Psi_{32}}{\partial \upsilon}\right)\Bigg|_{\upsilon=\gamma Q}, 0 \le \theta \le \pi/2 \tag{3.13}$$

The solution of problem (3.2)–(3.5), (3.8), (3.11)–(3.13) has the form

$$\Psi_1(\upsilon, \theta) = \sum A_n(\kappa\tilde{\upsilon})^{2n}\sin 2n\theta$$

$$\Psi_{31}(\upsilon, \theta) = \sum\left[B_{1n}\tilde{\upsilon}^{2n} + B_{2n}\left(\frac{\gamma}{\upsilon}\right)^{2n}\right]\sin 2n\theta$$

$$\Psi_{32}(\upsilon, \theta) = \sum\left[C_n\tilde{\upsilon}^{-2n} + 2B_{1n}\right]\sin 2n\theta$$

where

$$A_n = \frac{(-1)^{n+1}\alpha}{\pi(1-\alpha)nQ^{2n-1}a_n}, \quad B_{1n} = \frac{(-1)^{n+1}Q}{\pi n}$$

$$B_{2n} = \frac{(-1)^n(2n+1)}{2\pi nQ^{2n-1}a_n}, \quad C_n = \frac{(-1)^n Q}{\pi n}\left[1 + \frac{2n+1}{2Q^{4n}a_n}\right]$$

The coordinates $X_1(\theta)$, $Y_1(\theta)$ and $X_2(\theta)$, $Y_2(\theta)$ of the lines BF ($|\nabla v| = \gamma$) and CE ($|\nabla v| = \alpha\gamma$) (Fig. 1b) are specified by the equations

$$X_1(\theta) = 1 + \frac{1}{\pi}\left[\delta_1 - \frac{2Q^2\alpha}{1-\alpha}\sum g_n\cos(2n-1)\theta\right]$$
$$Y_1(\theta) = \frac{1}{\pi}\left[\delta_2 - \frac{2Q^2\alpha}{1-\alpha}\sum g_n\sin(2n-1)\theta\right]$$
$$X_2(\theta) = 1 - \frac{2}{\pi(1-\alpha)}\sum h_n\cos(2n-1)\theta$$
$$Y_2(\theta) = \frac{2}{\pi(1-\alpha)}\sum h_n\sin(2n-1)\theta \tag{3.14}$$

where

$$\delta_1 = \left(Q^2 - 1\right)\operatorname{arctg}\frac{2Q\cos\theta}{Q^2 - 1}, \quad \delta_2 = \frac{1}{2}\left(Q^2 + 1\right)\ln\frac{Q^2 + 2Q\sin\theta + 1}{Q^2 - 2Q\sin\theta + 1}$$

$$g_n = \frac{(-1)^{n+1}}{(2n-1)Q^{2n-1}(a_n - 3/2)}, \quad h_n = \frac{(-1)^{n+1}}{(2n-1)Q^{2n-1}a_n}$$

It can be established that the series obtained also converge absolutely.

When $Q = 1$ and $\alpha = ½$ we have

$$x_2(0) = 1 - \frac{4}{3}\pi, \quad y_2(\pi/2) = \frac{8}{3}\pi$$

We will now consider the case (Problem 2), when

$$g_9(\xi) = \begin{cases} 0, & 0 \le \xi < \gamma \\ [0, \gamma], & \xi = \gamma \\ \xi, & \xi > \gamma \end{cases} \tag{3.15}$$

In this case the characteristics of the exact solution are obtained by the jet theory method,[13] and also by transformation of the velocity hodograph (see Refs 1 and 12). The coordinates of the line on which the modulus of the pressure gradient has a constant value γ when $Q \ge 1$ are given by the equations

$$X_1(\theta) = 1 + \frac{1}{\pi}(\delta_1 - 2Q\cos\theta), \quad X_1(\theta) = \frac{1}{\pi}(\delta_2 - 2Q\sin\theta) \tag{3.16}$$

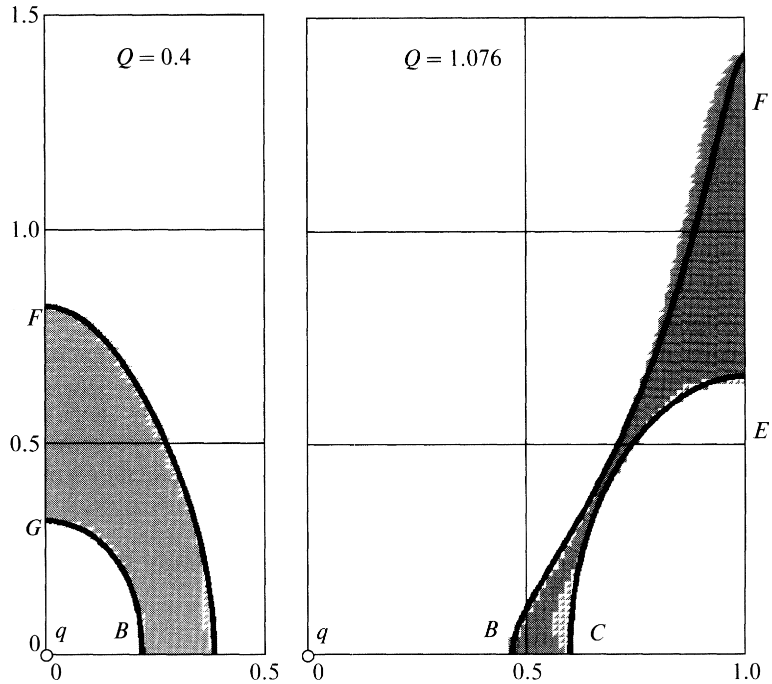

Fig. 3.

4. Results of numerical experiments

An iterative splitting method (2.2)–(2.4) was used to solve the problems considered in the previous section. In the numerical solution of the problems, the half-strip, shown in Fig. 1, was replaced by a finite region

$$(X, Y) = \{[0,1] \times [0, Z]\}, \quad Z \gg 1$$

on three parts of the boundary ($X=0$, $X=1$ and $Y=0$) of which the conditions $(v, \mathbf{n}) = 0$ are specified, while on the boundary Γ_∞ ($Y=Z$) "cutting off" infinity, a homogeneous Dirichlet condition $v = 0$ is specified. A bore hole with a flow rate q is situated at the point O. The dimensionless flow-rate parameter Q characterizes the seepage rate at infinity $v_\infty = Q\gamma$.

To construct a finite-dimensional approximation of the problem, we carried out a triangulation of the region obtained by uniform splitting of the sides Ω into n_1 and n_2 parts, a construction of triangles with diagonals parallel to the bisectrix of the first and third coordinate angles, and the use of the finite element method employing piecewise-linear functions on the triangles. The criterion of the ending of the iteration process was achieving a relative difference in the values of the approximate solution in neighbouring iterations of the specified accuracy $\varepsilon = 10^{-3}$. We chose the following values of the input parameters of the problem for the calculations: $\gamma = 1$ and $\alpha \in (0, 1)$ for Problem 1, and $\gamma = 1$ and $\alpha = 0$ for Problem 2, splitting of the region $n_1 = 20$ and $n_2 = 200$, $Z = 10$, and the values of the iteration parameters τ and ρ were varied from 0.1 to 2 in steps of 0.1.

The results of numerical experiments for Problem 1, when the function g_ϑ is defined by formula (3.1), are shown in Fig. 3 for $Q = 0.4$ ($Q \le \alpha$) and for $Q = 1.076$ ($Q \ge 1$). The curves BG and CF in the left-hand part of Fig. 3 are the boundaries of the zone Ω_γ, constructed using formulae (3.10). The curves BF and CE on the right-hand part of Fig. 3 are the boundaries of the zone Ω_γ, constructed using formulae (3.14).

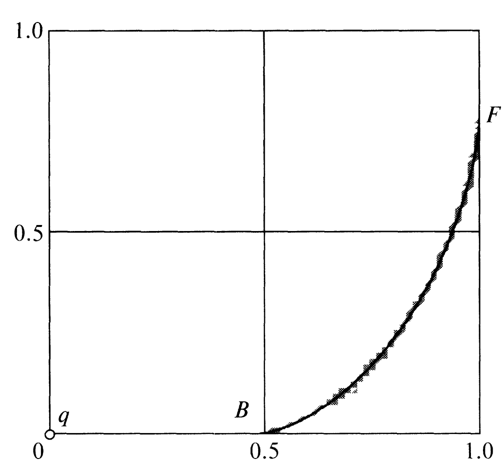

Fig. 4.

Regions in which the modulus of the gradient of the approximate solution (and actually approximate values of the modulus $\dot{y} = \Lambda p = \nabla p$) differs from γ by an amount 5×10^{-4} are distinguished by the darker shading. As a result of the numerical solution of this problem it was established that, when the flow rate of the bore hole q increases, and correspondingly the parameter Q also, the dimensions of the stagnation zone Ω_γ decrease.

The results of the numerical experiments for Problem 2, when the function g_ϑ is defined by formula (3.15), are shown in Fig. 4 for $Q = 1.4$. The curve BF is the boundary of the zone Ω_γ, constructed using formulae (3.16).

Hence, numerical experiments, carried out for model problems, have shown good agreement with the results obtained by approximate and analytical methods. This confirms the effectiveness of the iteration method considered, which enables us to obtain approximate values of the solution of the seepage problem for a multivalued seepage law in the case of an arbitrary region Ω and when conditions (1.4)–(1.6) are satisfied.

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